Name (IN CAPITALS): **Version #1**

Instructor: Isaac Newton

Math 10550 Exam 1
Sept. 19, 2023.

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off all cellphones, smartwatches and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all 19 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)

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Multiple Choice	________
11.	________
12.	________
13.	________
14.	________
Total	________

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3.	(a)	(b)	(c)	(d)	(e)
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8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
11.	________
12.	________
13.	________
14.	________
Total	________

Multiple Choice

1.(6pts) If $f(x) = \sin(x)$ and $g(x) = x - \frac{\pi}{2}$, which of the following is the graph of

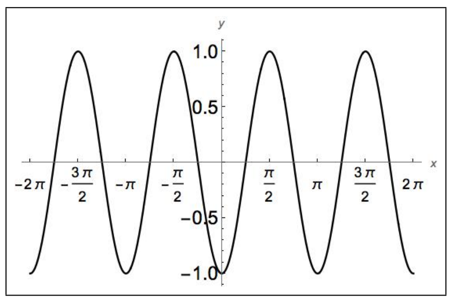
$$y = f(g(2x)) = (f \circ g)(2x)?$$

(Make sure you look carefully at the labels on both axes.)

Solution: We have $g(2x) = 2x - \frac{\pi}{2}$, so that $f(g(2x)) = f(2x - \frac{\pi}{2}) = \sin(2x - \frac{\pi}{2})$. If we ignore the $-\frac{\pi}{2}$ part, as x ranges from 0 to π, $\sin(2x)$ ranges from $\sin(0)$ to $\sin(2\pi)$, completing the period of the sine function. That is, for x from 0 to π, the graph of $y = \sin(2x)$ starts at zero (for $x = 0$), goes up to 1 (at $x = \frac{\pi}{4}$), back at zero (at $x = \frac{\pi}{2}$), down to -1 (at $x = \frac{3\pi}{4}$) and back to 0 (at $x = \pi$). Now recall that the graph of a function $h(x - c)$ is the graph of $h(x)$ shifted c units to the right. If we let $h(x) = \sin(2x)$, then

$$h(x - \frac{\pi}{4}) = \sin(2(x - \frac{\pi}{4})) = \sin(2x - \frac{\pi}{2}).$$

Therefore, the graph of $\sin(2x - \frac{\pi}{2})$ is the graph of $\sin(2x)$ shifted $\frac{\pi}{4}$ units to the right. The answer is:

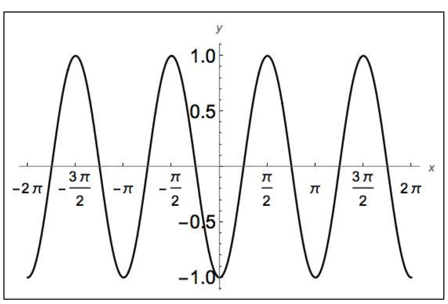


We can double check by plugging some values of x:

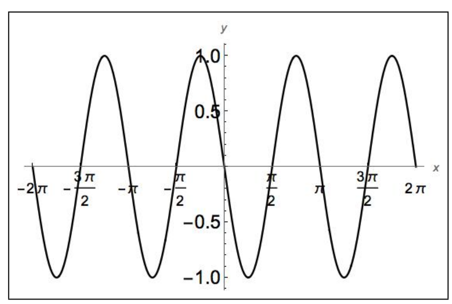
$$x = 0 \implies f(g(2 \cdot 0)) = \sin(-\frac{\pi}{2}) = -1$$

$$x = \frac{\pi}{2} \implies f(g(2 \cdot \frac{\pi}{2})) = \sin(\pi - \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1.$$

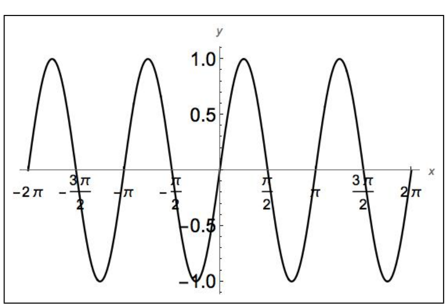
(a)



(b)



(c)



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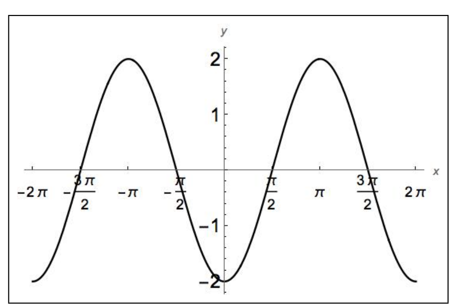
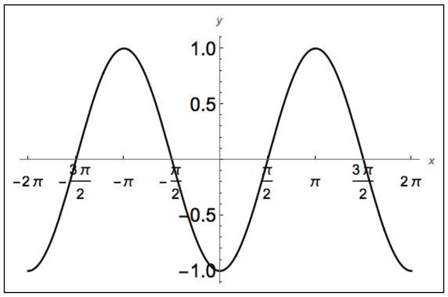


3. Initials: ____________

(e) 

2.(6pts) A stone dropped from a height of 12 feet on the planet Zluto at time $t = 0$ seconds has height at time t (seconds) given by

$$h(t) = 12 - t^2 - t, \quad t \geq 0$$

What is the average speed of the stone on the time interval $[0, t_1]$ where t_1 is the time at which the ball first hits the surface of the planet.
Note: Speed = |velocity|.

Solution: First we find t_1, which is the value at which $h(t_1) = 0$. Since

$$h(t) = -(t^2 + t - 12) = -(t+4)(t-3),$$

we have that $h(t) = 0$ at $t = 3$ and $t = -4$. Since $t \geq 0$, we conclude $t_1 = 3$.

The average speed is the absolute value of the slope of the secant line through the points on the curve at $t = 0$ and $t = 3$, that is,

$$\left|\frac{\Delta h(t)}{\Delta t}\right| = \left|\frac{h(3) - h(0)}{3 - 0}\right| = \left|\frac{0 - 12}{3}\right| = 4 \text{ ft/s}.$$

(a) 4 ft/s (b) 3 ft/s (c) 12 ft/s (d) 7 ft/s (e) 1 ft/s

3.(6pts) Consider the piecewise defined function:

$$p(x) = \begin{cases} x^2 - 1 & \text{if } -\infty < x < -1 \\ 2x + 1 & \text{if } -1 \leq x < 1 \\ x - 2 & \text{if } 1 \leq x < \infty \end{cases}$$

Which of the following statements are true?

(1) $\lim_{x \to -1^-} p(x) = \lim_{x \to -1^+} p(x)$.

(2) $\lim_{x \to 1^-} p(x) = p(1)$.

(3) The domain of $p(x)$ is all real numbers.

(4) $p(p(1)) = 0$.

Solution: We check each of the statements:

(1) *False* because

$$\lim_{x \to -1^-} p(x) = \lim_{x \to -1^-} x^2 - 1 = 0$$

while

$$\lim_{x \to -1^+} p(x) = \lim_{x \to -1^+} 2x + 1 = -1.$$

(2) *False* because $\lim_{x \to 1^-} 2x - 1 = 3$ and $p(1) = 1 - 2 = -1$.

(3) *True.* There are no possible zero denominators or negative square roots.

(4) *False.* From (2), $p(1) = -1$, so that $p(p(1)) = p(-1) = 2(-1) + 1 = -1$.

Therefore, only statement (3) is true.

(a) Only statement (3) is true.

(b) Only statements (1), (3) and (4) are true.

(c) None of the statements are true.

(d) Only statements (1), (2) and (3) are true.

(e) All of the statements are true.

4.(6pts) Evaluate the following limit:

$$\lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x-3}.$$

Solution: Note first that $\lim_{x\to 3} x^2+7=0$ and $\lim_{x\to 3} x-3=0$. Thus, we rationalize to evaluate the limit:

$$\begin{aligned}\frac{\sqrt{x^2+7}-4}{x-3} &= \frac{(\sqrt{x^2+7}-4)(\sqrt{x^2+7}+4)}{(x-3)(\sqrt{x^2+7}+4)}\\ &= \frac{x^2+7-16}{(x-3)(\sqrt{x^2+7}+4)}\\ &= \frac{(x-3)(x+3)}{(x-3)(\sqrt{x^2+7}+4)}\\ &= \frac{x+3}{\sqrt{x^2+7}+4}.\end{aligned}$$

Then

$$\lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x-3} = \lim_{x\to 3} \frac{x+3}{\sqrt{x^2+7}+4} = \frac{3+3}{\sqrt{3^2+7}+4} = \frac{6}{8} = \frac{3}{4}.$$

(a) $\frac{3}{4}$ (b) 6 (c) $\frac{4}{3}$ (d) ∞ (e) $\frac{1}{6}$

5.(6pts) Compute

$$\lim_{x\to 2^-} \frac{x^2-4}{x^2-4x+4}.$$

Solution: Notice that

$$\frac{x^2-4}{x^2-4x+4} = \frac{(x-2)(x+2)}{(x-2)^2} = \frac{x+2}{x-2}.$$

Since $\lim_{x\to 2^-} x+2=4$, while $\lim_{x\to 2^-} x-2=0$, we know that $\frac{x+2}{x-2}$ tends to infinity <u>in absolute value</u> as x approaches 2 from the left. As $x\to 2^-$, we have that $x+2$ is positive, while $x-2$ is negative (since x approaches 2 from the left, so $x<2$). Therefore,

$$\lim_{x\to 2^-} \frac{x^2-4}{x^2-4x+4} = \lim_{x\to 2^-} \frac{x+2}{x-2} = -\infty.$$

(a) $-\infty$ (b) 0

(c) ∞ (d) 1

(e) Does not exist and is not $+\infty$ or $-\infty$

6.(6pts) Let $f(x) = x\cos(x)$. If you were trying to calculate $f'(\pi)$ using the limit definition of the derivative, which of the following would give the correct expression?

Note: Pay careful attention to the $h \to ...$ part.

Solution: We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Therefore,

$$\begin{aligned} f'(\pi) &= \lim_{h\to 0} \frac{f(\pi+h) - f(\pi)}{h} \\ &= \lim_{h\to 0} \frac{(\pi+h)\cos(\pi+h) - \pi\cos(\pi)}{h} \\ \text{(Since } \cos(\pi) = -1\text{)} \qquad &= \lim_{h\to 0} \frac{(\pi+h)\cos(\pi+h) + \pi}{h} \end{aligned}$$

(a) $\lim_{h\to 0} \frac{(\pi+h)\cos(\pi+h)+\pi}{h}$

(b) $\lim_{h\to 0} \frac{(\pi+h)\cos(\pi+h)-\pi}{h}$

(c) $\lim_{h\to 0} \frac{(\pi+h)\cos(\pi+h)+1}{h}$

(d) $\lim_{h\to 0} \frac{\pi\cos(\pi)-h\cos(h)-\pi}{h}$

(e) $\lim_{h\to \pi} \frac{\pi\cos(\pi)-h\cos(h)+\pi}{h}$

7.(6pts) Let $f(x) = \dfrac{4}{\sqrt{x}} - \sqrt[3]{x}$. Which of the following gives $f'(1)$.

Solution: We can rewrite the function $f(x)$ as follows,

$$f(x) = 4x^{-\frac{1}{2}} - x^{\frac{1}{3}}.$$

By using the power rule for derivatives, we get that

$$\begin{aligned} f'(x) &= 4(-\frac{1}{2})x^{-\frac{1}{2}-1} - (\frac{1}{3})x^{\frac{1}{3}-1} \\ &= -2x^{-\frac{3}{2}} - \frac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{-2}{\sqrt[2]{x^3}} - \frac{1}{3\sqrt[3]{x^2}}. \end{aligned}$$

Thus,

$$\begin{aligned} f'(1) &= \frac{-2}{\sqrt{(1)^3}} - \frac{1}{3\sqrt[3]{(1)^2}} \\ &= -2 - \frac{1}{3} \\ &= -\frac{7}{3} \end{aligned}$$

(a) $-\dfrac{7}{3}$ (b) $-\dfrac{5}{3}$ (c) 3 (d) $-\dfrac{13}{3}$ (e) $\dfrac{5}{3}$

8.(6pts) If

$$f(x) = \frac{x^2 + 2x + 1}{x^{10} + x},$$

find $f'(x)$.

Solution: We apply the quotient rule for derivatives to the rational function $f(x)$ to get the following

$$\begin{aligned} f'(x) &= \frac{(x^{10} + x)(x^2 + 2x + 1)' - (x^2 + 2x + 1)(x^{10} + x)'}{(x^{10} + x)^2} \\ &= \frac{(x^{10} + x)(2x + 2) - (x^2 + 2x + 1)(10x^9 + 1)}{(x^{10} + x)^2} \end{aligned}$$

(a) $\dfrac{(x^{10} + x)(2x + 2) - (x^2 + 2x + 1)(10x^9 + 1)}{(x^{10} + x)^2}$

(b) $(x^{10} + x)(2x + 2) + (x^2 + 2x + 1)(10x^9 + 1)$

(c) $\dfrac{(x^2 + 2x + 1)(10x^9 + 1) - (x^{10} + x)(2x + 2)}{(x^{10} + x)^2}$

(d) $\dfrac{2x + 2}{10x^9 + 1}$

(e) $\dfrac{-(2x + 2)}{(x^{10} + x)^2}$

9.(6pts) Compute
$$\lim_{x\to 0} \frac{\sin(3x)}{\cos(x+\pi)\sin(2x)}.$$

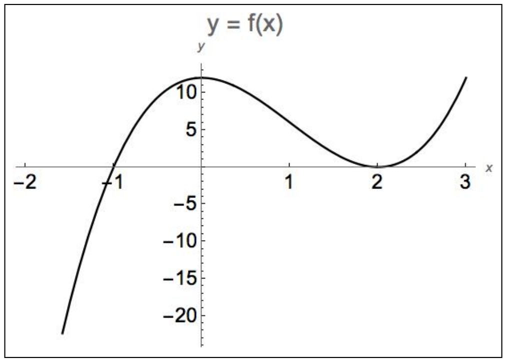
Solution: We know that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Moreover, $\lim_{x\to 0} \frac{x}{\sin(x)} = 1$. Using the above we can rewrite the expression in this problem as follows:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(3x)}{(\cos(x+\pi)\sin(2x)} &= \lim_{x\to 0} \left(\frac{\sin(3x)}{\cos(x+\pi)\sin(2x)} \cdot \frac{3x}{3x} \cdot \frac{2x}{2x} \right) \\
&= \lim_{x\to 0} \left(\frac{\sin(3x)}{3x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{3x}{2x\cos(x+\pi)} \right) \\
&= \left(\lim_{x\to 0} \frac{\sin(3x)}{3x} \right) \left(\lim_{x\to 0} \frac{2x}{\sin(2x)} \right) \left(\lim_{x\to 0} \frac{3x}{2x\cos(x+\pi)} \right) \\
&= (1)(1) \left(\frac{3}{2\cos(\pi)} \right) \\
&= -\frac{3}{2}.
\end{aligned}$$

(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$ (c) ∞ (d) $-\frac{2}{3}$ (e) 0

 Initials: ____________

10.(6pts) The graph of $f(x)$ is shown below:

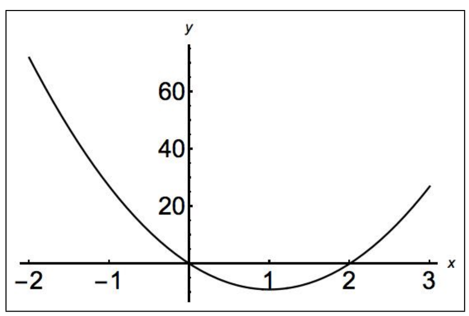


Which of the following is the graph of $f'(x)$?

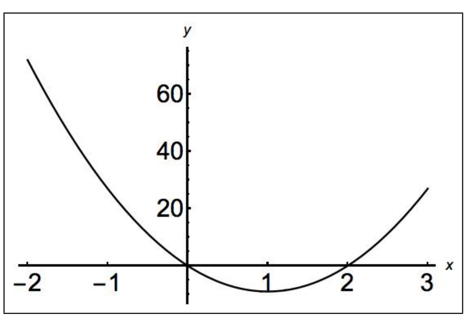
Solution: By looking at how the graph of $f(x)$ curves we can check whether $f'(x) > 0, f'(x) < 0$, or $f'(x) = 0$. We will just write the values of x, in terms of intervals, where the above three conditions are satisfied:

- $f'(x) > 0$ at the intervals $(-2, 0) \cup (2, 3]$.
- $f'(x) = 0$ at the points $x = 0$ and $x = 2$. Hence, the graph of $f'(x)$ crosses the x-axis at the points $(0, 0)$ and $(2, 0)$.
- $f'(x) < 0$ at the interval $(0, 2)$.

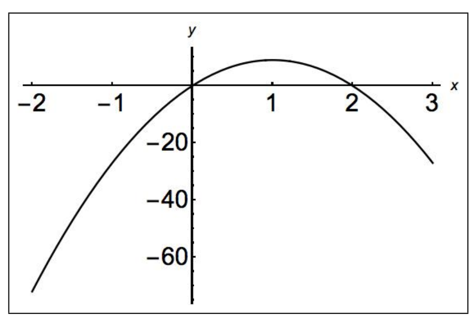
Therefore, the graph of $f'(x)$ is



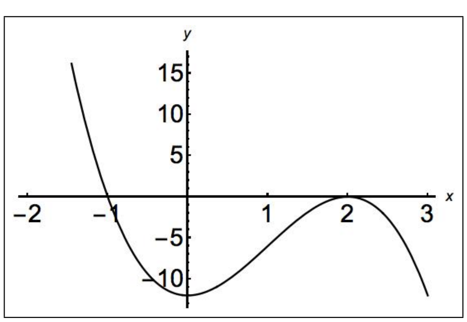
(a)



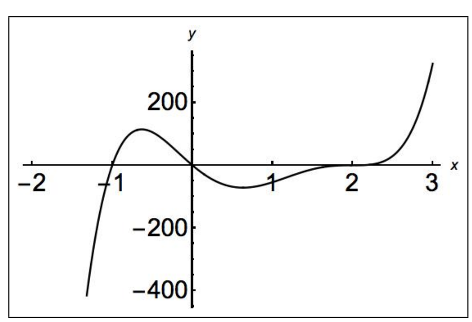
(b)



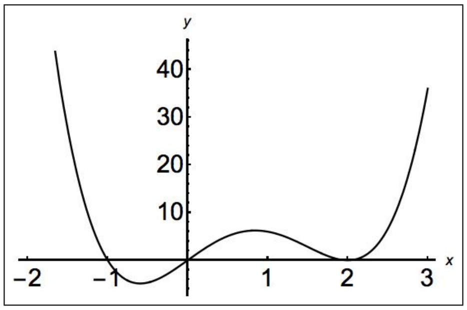
(c)



(d)



12. Initials: ____________

(e) 

Partial Credit

For full credit on partial credit problems, make sure you justify your answers.

11.(12pts) Show that there is at least 1 solution to the equation

$$x^3 + 4x - 2 = 0$$

in the interval $0 \le x \le 1$. Justify your answer and identify any theorems you use.

Solution: Let $f(x) = x^3 + 4x - 2$. Note that f is a continuous function because it is a polynomial. Moreover, $f(0) = -2$, while $f(1) = 3$. Since $f(0) < 0 < f(1)$ and f is continuous, we invoke the Intermediate Value Theorem to conclude that there exists some c between 0 and 1 such that $f(c) = 0$.

12.(14pts) (a) Find the derivative of

$$f(x) = \frac{1}{x+3}$$

using <u>the limit definition</u> of the derivative following the steps below.
(a1) Write down the limit definition of the derivative of a function $f(x)$;

$$f'(x) =$$

Solution: The limit definition of the derivative of a function $f(x)$ is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

(a2) Rewrite the expression for the limit in part 1 using the function $f(x) = \dfrac{1}{x+3}$ given in the problem.

$$f'(x) =$$

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \end{aligned}$$

(a3) Use algebra to simplify your expression for the limit from part (a2). (Please include your limit symbols in the calculation.) **Solution:**

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h+3} - \frac{1}{x+3}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+3) - (x+h+3)}{(x+h+3)(x+3)}\right) \\ &= \lim_{h\to 0} \frac{-h}{h(x+h+3)(x+3)} \\ &= \lim_{h\to 0} -\frac{1}{(x+h+3)(x+3)} \end{aligned}$$

(a4) Evaluate the limit from part (a3) to find $f'(x)$. **Solution:**

$$\begin{aligned} f'(x) &= \lim_{h\to 0} -\frac{1}{(x+h+3)(x+3)} \\ &= -\frac{1}{(x+3)(x+3)} \\ &= -\frac{1}{(x+3)^2}. \end{aligned}$$

(b) use your answer from part (a) to find the equation of the tangent line to the graph of $y = f(x)$ at $x = 1$.

Solution: To calculate the equation of the tangent line to the graph at $x = 1$, we first calculate the slope of the tangent line, which is

$$\begin{aligned} f'(1) &= -\frac{1}{(1+3)^2} \\ &= -\frac{1}{16}. \end{aligned}$$

By the point-slope formula of a line using the point $(1, f(1)) = (1, \frac{1}{4})$, the equation of the tangent line is

$$\begin{aligned} y - \frac{1}{4} &= -\frac{1}{16}(x-1) \\ y &= -\frac{1}{16}x + \frac{5}{16}. \end{aligned}$$

13.(12pts) (a) Let $f(x) = \cos(\sqrt{x^5+\pi^2}) + 1$. Find $f'(x)$.

Solution: The derivative of $f(x)$ is the following

$$\begin{aligned} f'(x) &= -\sin(\sqrt{x^5+\pi^2})(\frac{d}{dx}(\sqrt{x^5+\pi^2}) + \frac{d}{dx}(1) \\ &= -\sin(\sqrt{x^5+\pi^2})(\frac{1}{2\sqrt{x^5+\pi^2}})(\frac{d}{dx}(x^5+\pi^2)) + 0 \\ &= -\sin(\sqrt{x^5+\pi^2})(\frac{1}{2\sqrt{x^5+\pi^2}})(5x^4) \\ &= -\frac{5x^4\sin(\sqrt{x^5+\pi^2})}{2\sqrt{x^5+\pi^2}}. \end{aligned}$$

(b) Consider the piecewise defined function

$$p(x) = \begin{cases} x+c & \text{if } -\infty < x < 0 \\ \cos(\sqrt{x^5+\pi^2}) + 1 & \text{if } 0 \le x < \infty \end{cases}$$

where c is a constant.

Is it possible to find a value of c such that $p(x)$ is continuous at $x = 0$? Please explain your answer and if such a value of c exists, find it.

Solution: For the function to be continuous at $x = 0$ we just need to make sure that the limits of $p(x)$ as x approaches 0 from the left and right are the same. We have that

$$\lim_{x\to 0^-} p(x) = \lim_{x\to 0^-} (x+c) = c,$$

and

$$\begin{aligned} \lim_{x\to 0^+} p(x) &= \lim_{x\to 0^+} (\cos(\sqrt{x^5+\pi^2}) + 1) \\ &= \cos(\sqrt{\pi^2}) + 1 \\ &= \cos(\pi) + 1 \\ &= -1 + 1 \\ &= 0. \end{aligned}$$

Then for continuity we must have

$$\lim_{x\to 0^-} p(x) = \lim_{x\to 0^+} p(x) \implies c = 0.$$

We conclude that for $c = 0$, $p(x)$ is continuous at $x = 0$.

(c) Is it possible to find a value of c such that the function $p(x)$ defined in part (b) is differentiable at $x = 0$? Please explain your answer and if such a value of c exists, find it.
Solution *Solution.* For the function to be diffetentiable at $x = 0$ it must be continuous at $x = 0$. From part (b) we get that $c = 0$, so that

$$p(x) = \begin{cases} x & \text{if } -\infty < x < 0 \\ \cos(\sqrt{x^5 + \pi^2}) + 1 & \text{if } 0 \leq x < \infty \end{cases}$$

Then, for the function to be differentiable at $x = 0$ the left and right derivatives must agree, that is,

$$\lim_{h \to 0^-} \frac{p(x+h) - p(x)}{h} = \lim_{h \to 0^+} \frac{p(x+h) - p(x)}{h}.$$

However, $\lim_{h \to 0^-} \frac{p(x+h)-p(x)}{h}$ is the derivative of x at $x = 0$, thus

$$\lim_{h \to 0^-} \frac{p(x+h) - p(x)}{h} = 1.$$

On the other hand, $\lim_{h \to 0^+} \frac{p(x+h)-p(x)}{h}$ is the derivative of $\cos(\sqrt{x^5 + \pi^2}) + 1$ at $x = 0$, which from part (a) is

$$\lim_{h \to 0^+} \frac{p(x+h) - p(x)}{h} = -\frac{5(0)^4 \sin(\sqrt{(0)^5 + \pi^2})}{2\sqrt{(0)^5 + \pi^2}} = 0.$$

Since $1 \neq 0$, it does *not* exist a value of c for which $p(x)$ is differentiable at $x = 0$.

14.(2pts) You will be awarded these two points if you write your name in CAPITALS on the front page and you mark your answers on the front page with an X through your answer choice like so: ~~(a)~~ (not an O around your answer choice) . You may also use this page for

ROUGH WORK